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First Integrals and Parametric Solutions for Equations Integrable Through Lie Symmetries

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Abstract

We present here the explicit parametric solutions of second order differential equations invariant under time translation and rescaling and third order differential equations invariant under time translation and the two homogeneity symmetries. The computation of first integrals gives in the most general case, the parametric form of the general solution. For some polynomial functions we obtain a time parametrisation quadrature which can be solved in terms of "known" functions.

1 Introduction

Many works deal with the problem of integrability of ordinary differential equations. Although the concept of "integrability" is delicate, at least four "routes" contribute to explore it. The first one consists of expressing the general solution of an ordinary differential equation in terms of "known" functions; the second deals with the Painlevé analysis of the equation's singularities; the third uses the Lie symmetry method and the last one is characterized by the existence of invariants, called first integrals when the independent variable does not explicitly appear. Usually a N^{th} order *autonomous* ordinary differential equation is said to be "integrable" if it possesses $N-1$ functionally independent first integrals. In this paper we consider the class of ordinary differential equations invariant under time translation (G_1) and rescaling (G_2), which frequently occur in the modelling of natural phenomena. We write the generators as

$$G_1 = \partial_t, \qquad G_2 = -qt\partial_t + x\partial_x \qquad (q \in \mathbb{Z}), \tag{1}$$

where t is the independent variable and x the dependent variable. The symmetries G_1 and G_2 constitute a representation of Lie's Type III two-dimensional algebra [1]. The general form of the second order ordinary differential equation invariant under the action of the two symmetries G_1 and G_2 is

$$\ddot{x} + x^{2q+1} f(\xi) = 0, \qquad \xi = \frac{\dot{x}}{x^{q+1}} \qquad (q \in \mathbb{Z}), \tag{2}$$

where $\ddot{x}$ and $\dot{x}$ represent respectively $\mathrm{d}^2x/\mathrm{d}t^2$ and $\mathrm{d}x/\mathrm{d}t$.

We also treat the class of third order equations possessing the three symmetries associated with the generators

$$G_1 = \partial_t, \qquad G_{21} = t\partial_t, \qquad G_{22} = x\partial_x. \tag{3}$$

The second and third of these, G_{21} and G_{22}, are called homogeneity symmetries [2]. The general form of a third order ordinary differential equation possessing the symmetries (3) is

$$\dddot{x} + \frac{\dot{x}^3}{x^2} F(\rho) = 0, \qquad \rho = \frac{x\ddot{x}}{\dot{x}^2} \cdot \tag{4}$$

In (4) F is an arbitrary function of its argument and ρ is the second order differential invariant common to G_1, G_{21} and G_{22}. Equation (4) is representative of the class of third order ordinary differential equations invariant under the three-dimensional algebra with the single nonzero Lie Bracket, $[G_1, G_2] = G_1$, denoted by $A_1 \oplus A_2$ in the Mubarakzyanov classification scheme [3, 4, 5]. We note that the two-dimensional algebra A_2 is of Lie's Type IV since now $G_{21} = tG_1$. We also note that (2) is naturally connected to (4) by the Riccati transformation $x = (1 + 2/q)^{1/q} (\dot{u}/u)^{1/q}$.

This paper has two practical goals, *viz* firstly to obtain explicit expressions for the first integrals needed to declare (2) and (4) integrable and secondly to use these first integrals to push the analytic computations as far as possible in order to obtain parametric solutions taking the form of quadratures which can be performed in closed form when the functions $f(\xi)$ (in (2)) and $F(\rho)$ (in (4)) are polynomials.

The paper is divided as follows. In the second section we treat the case of the second order ordinary differential equation (2). In the third section we show how to obtain many first integrals (with of course only two independent) for the third order ordinary differential equation (4) and how from these first integrals we deduce the parametric form of the global solution. For functions f and F polynomials of at most third degree we present an explicit solution of the time parametrisation quadrature. In the fourth section we present our conclusions.

2 Parametric solution for a class of Sordinary differential equation

The important result is the following reduced form of (2) [6, equation (8)]

$$\frac{\mathrm{d}x}{x} + \frac{\xi \mathrm{d}\xi}{(q+1)\xi^2 + f(\xi)} = 0, \tag{5}$$

which provides the first integral of (2). We write

$$D(\xi) = f(\xi) + (q+1)\xi^2 \tag{6}$$

and consequently

$$x = x_0 \exp\left[-\int_{\xi_0}^{\xi} \frac{u\mathrm{d}u}{D(u)}\right] \tag{7}$$

In (7) the values x_0 and ξ_0 refer to the values of x and ξ at time $t = 0$ which can always be taken as initial time because of the symmetry G_1. The parametric solution of $\dot{x}$ which is needed to compute the time scale with $\mathrm{d}t = \mathrm{d}x/\dot{x}$ is given by $\dot{x} = \xi x^{q+1}$. Taking (7) into account

$$\dot{x} = x_0^{q+1}\xi \exp\left[-(q+1)\int_{\xi_0}^{\xi} \frac{u\mathrm{d}u}{D(u)}\right]. \tag{8}$$

For polynomial $f(\xi)$ the integrations in (7) and (8) can be performed in closed form and for polynomials of degree two the integral giving $t(\xi)$ can be expressed in terms of hypergeometric functions.

3 First integrals and parametric solution for a class of third order ordinary differential equations

It was shown [6, equation (6)] that a third order ordinary differential equation invariant under time translation and rescaling symmetries can be written as

$$\dddot{x} + x^{3q+1} f(\xi,\eta) = 0, \qquad \xi = \frac{\dot{x}}{x^{q+1}}, \qquad \eta = \frac{\ddot{x}}{x^{2q+1}}, \tag{9}$$

where f is an arbitrary function of its arguments. The connection between the two forms as given by (9) and (4) (i.e. the relation between f and F) is easy to establish. Firstly the argument ρ is equal to η/ξ^2 for all q. The identification of the two terms in front of f and F gives

$$f(\xi,\eta) = \xi^3 F\left(\frac{\eta}{\xi^2}\right). \tag{10}$$

Moreover it was shown [6, equation (89)] that by applying the two symmetries G_1 and G_2 to (9) we obtain

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{f(\xi,\eta) + (2q+1)\xi\eta}{(q+1)\xi^2 - \eta}. \tag{11}$$

We introduce (10) in (11), change the variable ξ to $z = \frac{1}{2}\xi^2$ and reformulate (11) with z and ρ to obtain separation of variables and the following first integral

$$\frac{\mathrm{d}z}{z} = \frac{2(q+1-\rho)}{F(\rho) + 2\rho^2 - \rho}\mathrm{d}\rho. \tag{12}$$

Consequently we can replace (4) by (12) where now q can be arbitrarily chosen. Now q being chosen arbitrarily we obtain as many first integrals as we want, but, of course, only two will be independent. For polynomial $F(\rho)$, q can be chosen to exhibit the simplest possible form for these first integrals. In order to illustrate our procedure we consider two polynomial functions $F(\rho)$. Taking first a quadratic polynomial we write

$$F(\rho) + 2\rho^2 - \rho = \gamma(\rho - \alpha_1)(\rho - \alpha_2) \qquad (\forall\, \gamma, \alpha_1, \alpha_2 \in \mathbb{R} \text{ and } \alpha_1 \neq \alpha_2) \tag{13}$$

(the case $\alpha_1 = \alpha_2$ is similar). Now we take in (12) $q_1 = \alpha_1 - 1$. Introducing (13) in (12) we obtain after a simple integration the first integral

$$I_1 = z_1^{\gamma/2}(\rho - \alpha_2) = \left(\frac{\dot{x}}{x^{\alpha_1}}\right)^{\gamma}(\rho - \alpha_2). \tag{14}$$

To find a second simple first integral, we take $q_2 = \alpha_2 - 1$ and obtain

$$I_2 = z_2^{\gamma/2}(\rho - \alpha_1) = \left(\frac{\dot{x}}{x^{\alpha_2}}\right)^{\gamma}(\rho - \alpha_1). \tag{15}$$

So far we have obtained two rather simple first integrals and the ordinary differential equation is now integrable. Moreover the time parametrisation of the trajectory is obtained by the elimination of ρ between I_1 (14) and I_2 (15). This gives

$$\dot{x} = \left(\frac{I_2 x^{\alpha_2\gamma} - I_1 x^{\alpha_1\gamma}}{\alpha_2 - \alpha_1}\right)^{1/\gamma} \tag{16}$$

and implies the time parametrisation quadrature

$$t = (\alpha_2 - \alpha_1)^{1/\gamma} \int \frac{\mathrm{d}x}{\left(I_2 x^{\alpha_2\gamma} - I_1 x^{\alpha_1\gamma}\right)^{1/\gamma}}\,. \tag{17}$$

Not surprisingly the performance of the quadrature in (17) is generally not possible in closed form. We turn now to an $F(\rho)$ which is a third degree polynomial in ρ. We write

$$\begin{aligned} &F(\rho) + 2\rho^2 - \rho = \gamma(\rho - \alpha_1)(\rho - \alpha_2)(\rho - \alpha_3), \\ &(\forall\, \gamma, \alpha_i \in \mathbb{R} \text{ and } \ \forall\, i \neq j,\ \alpha_i \neq \alpha_j). \end{aligned} \tag{18}$$

(If at least two roots α_i are equal, the problem is quite similar). Taking respectively $q_i = \alpha_i - 1$, $(i = 1, 2, 3)$, in (12) and after a direct integration we find the first integrals

$$\begin{aligned} &\frac{\dot{x}}{x^{\alpha_1}}^{\gamma(\alpha_2-\alpha_3)}(\rho - \alpha_2)(\rho - \alpha_3)^{-1} = I_1 \\ &\frac{\dot{x}}{x^{\alpha_2}}^{\gamma(\alpha_3-\alpha_1)}(\rho - \alpha_3)(\rho - \alpha_1)^{-1} = I_2 \\ &\frac{\dot{x}}{x^{\alpha_3}}^{\gamma(\alpha_1-\alpha_2)}(\rho - \alpha_1)(\rho - \alpha_2)^{-1} = I_3. \end{aligned} \tag{19}$$

The relation

$$I_1 I_2 I_3 = 1 \tag{20}$$

proves that the three first integrals are not independent. Thus any pair of (19) provides the two independent first integrals needed to declare the ordinary differential equation integrable.

We return now to the general problem. From (12) writing $q+1=\alpha_i$, $i=1,2$, we obtain

$$\frac{\dot{x}^2}{x^{2\alpha_i}} = \frac{\dot{x}_0^2}{x_0^{2\alpha_i}} \exp\left[\int_{\rho_0}^{\rho} \frac{2(\alpha_i-\rho)}{D(\rho)}\mathrm{d}\rho\right], \qquad i=1,2, \tag{21}$$

where $D(\rho)=F(\rho)+2\rho^2-\rho$. Firstly we eliminate $\dot{x}$ between the two equations (21) to give

$$x^{2(\alpha_2-\alpha_1)} = x_0^{2(\alpha_2-\alpha_1)} \exp\left[\int_{\rho_0}^{\rho} \frac{2(\alpha_1-\alpha_2)}{D(\rho)}\mathrm{d}\rho\right] \tag{22}$$

and consequently

$$x = x_0 \exp\left[-\int_{\rho_0}^{\rho} \frac{\mathrm{d}u}{D(u)}\right]. \tag{23}$$

Then we eliminate x to obtain $\dot{x}$

$$\dot{x} = \dot{x}_0 \exp\left[-\int_{\rho_0}^{\rho} \frac{u\mathrm{d}u}{D(u)}.\right] \tag{24}$$

Relations (23) and (24) give the parametric solution (with $\mathrm{d}t=\mathrm{d}x/\dot{x}$). Although (23) and (24) can be obtained directly (computing $\dot{\rho}$ and $\dot{\rho}\rho$) the rather simple expression for the two first integrals is obtained by selecting $q_i+1=\alpha_i$, $i=1,2$, in (12), α_1 and α_2 being two roots of $D(\rho)=0$. Finally we observe that the computation of the leading order term in the Painlevé analysis with a singularity in $(t-t_0)^{-1/q}$ gives the equation

$$(1+q)(1+2q)+F(1+q)=0. \tag{25}$$

If we write $\alpha=1+q$, this last equation becomes

$$F(\alpha)+2\alpha^2-\alpha=D(\alpha)=0. \tag{26}$$

This confirms the asymptotic character of some of these singular solutions since we recover the divergence of x in (23) for exactly the same value $q+1=x\ddot{x}/\dot{x}^2$ when $x\sim(t-t_0)^{-1/q}$.

4 Conclusion

This work deals with the possibility of exhibiting explicit first integrals and parametric solutions for second and third order ordinary differential equations possessing the necessary number of symmetries to be formally integrable. Quite naturally the parameters allowing these simple parametric solutions are the two differential invariants of the equation, namely ξ for the second order equation and ρ for the third order equation. For functions $f(\xi)$ and $F(\rho)$ polynomial the integrals giving x and $\dot{x}$ can be performed in closed form and for quadratic $f(\xi)$ the integral giving t is obtained in terms of hypergeometric functions. Finally for pathological functions f and F our method minimises numerical calculations. Moreover for the third order equation the main point is indeed the identity with the two homogeneity symmetries and rescaling with an arbitrary parameter q which makes our method systematic. For the obtention of first integrals, another aspect of these results is that they are at the same time connected and complementary to the Painlevé analysis.

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